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ATOMS Index Interest Rates Model
Maximum Smooth Forward Curves

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Introduction

There is general agreement now among academics and market practitioners that the Heath-Jarrow-Morton (HJM) approach to term structure modeling is the most elegant of all term structure models. For pricing and risk managing interest rate related contingent claims, the HJM model has many desirable features: (1) it's automatically arbitrage free by construction--there's no need to calibrate the model to the current yield curve, (2) yields can be made strictly positive in a wide range of settings, (3) volatility parameters are closely related to market volatilities such as historical yield volatilities or implied volatilities from interest rate options, and (4) it can accommodate arbitrary number of factors in a straight forward way. However, since HJM models the dynamics of the yield curve directly, the dynamics of individual rates are generally non-Markov[1]. It's therefore difficult to apply HJM to a PDE or tree based pricing framework. Monte Carlo/simulation based approach under HJM is generally the pricing method of choice among market pratitioners, which is called for in any case for pricing complex interest rate derivatives such mortgage-backed securities. In this note, we describe in simple terms how HJM model can be derived from simple no arbitrage conditions. We also describe in detail how model parameters are determined and how they are related to market observables. We will concentrate mostly on the *lognormal* specification of the HJM model and show how it's related to its close "cousin"--the recently developed Brace-Gatarek-Musiela (BGM) model.

Heath-Jarrow-Morton Model

Unlike the original Black-Scholes approach, *i.e.* coming up with a PDE which incorporates the market price of risk, the "modern" approach to derivatives relies on the existence of martingles[2] as the condtion of no-arbitrage. Here, we follow the martingale approach in our derivation of the HJM model.

For the term structure of interest rates, first let's define:

$$\beta(t) \equiv \exp\left(\int_0^t r(u)du\right)$$

where $r(t)$ is the instantaneous short interest rate, and also define

$B(t,T) \equiv$ price of a zero-coupon bond paying \$1 at time T

then it's simple to show[3] that

[1] A process that is Markov roughly means that it has no "memory" of the past. Under HJM, the forward yield curve is Markov, however. But this theoretical property of HJM does not really have any practical implications.
[2] A martingle is stochastic process that displays no drift. In other words, its future expectation is its current value.
[3] See, *e.g.*, Rebonato (1996).

$$\frac{B(t,T)}{\beta(t)}, \qquad 0 \le t \le T$$

is a martingle under the risk-neutral probability measure. In other words, under the risk-neutral measure *i.e.,* a probability measure under which one can not make risk-free profits by buying and selling bonds

$$B(0,T) = E^Q\left[\frac{B(t,T)}{\beta(t)}\right] \tag{1}$$

where we use $E^Q[]$ to indicate the expectation is taken under the risk-neutral measure[4].

Assume the stochastic differential equation (SDE) for the discount bond price is

$$dB(t,T) = \mu(t,T)B(t,T)dt + \nu(t,T)B(t,T)dW(t)$$

then

$$\begin{aligned} d\left(\frac{B(t,T)}{\beta(t)}\right) &= B(t,T)d\left(\frac{1}{\beta(t)}\right) + \frac{1}{\beta(t)}d(B(t,T)) \\ &= (\mu(t) - r(t))\frac{B(t,T)}{\beta(t)}dt + \nu(t,T)\frac{B(t,T)}{\beta(t)}dW(t) \end{aligned}$$

since $\frac{B(t,T)}{\beta(t)}$ is a martingale under the risk-neutral measure, *i.e.,* it should display no drift, therefore we have $\mu(t) = r(t)$. With this condition, we set out to find the drift condition for the SDE satisfied by the forward rate $f(t,T)$:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

where drift $\alpha(t,T)$, and volatility $\sigma(t,T)$ can depend on the forward rate itself.

Since

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

and

[4] Equation (1) is often referred to as the "tower" property of no-arbitrage, as it reminds one of a series of bond prices that are "chained" together.

$$d\left(-\int_t^T f(t,u)du\right) = f(t,t)dt - \int_t^T df(t,u)du$$
$$= r(t)dt - \int_t^T (\alpha(t,u)dt + \sigma(t,u)dW(t))du$$
$$= r(t)dt - \left(\int_t^T \alpha(t,u)du\right)dt - \left(\int_t^T \sigma(t,u)du\right)dW(t)$$

define $\alpha^*(t,T) \equiv \int_t^T \alpha(t,u)du$, $\sigma^*(t,T) \equiv \int_t^T \sigma(t,u)du$, and make use of Ito's lemma $dG(X) = G'(X)dX + \frac{1}{2}\sigma_X{}^2 G''(X)dt$ with $X \equiv -\int_t^T f(t,u)du$ and $G(X) \equiv \exp(X)$, one would then have

$$dB(t,T) = B(t,T)(r(t)dt - \alpha^*(t,T)dt - \sigma^*(t,T)dW(t)) + \tfrac{1}{2}(\sigma^*(t,T))^2 B(t,T)dt$$
$$= B(t,T)(r(t) - \alpha^*(t,T) + \tfrac{1}{2}(\sigma^*(t,T))^2)dt - \sigma^*(t,T)B(t,T)dW(t)$$

As we knew earlier that the risk-neutral drift of $B(t,T)$ is $r(t)$, which would then give us

$$\alpha^*(t,T) = \tfrac{1}{2}(\sigma^*(t,T))^2$$

or

$$\int_t^T \alpha(t,u)du = \tfrac{1}{2}\left(\int_t^T \sigma(t,u)du\right)^2 \qquad (2)$$

this is equivalent to

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du \qquad (3)$$

Equation (3) means that under the risk-neutral probability measure, the drift of the forward rate process is completely determined by the volatility function. The SDE for the forward rate process is then

$$df(t,T) = \sigma(t,T)\left(\int_t^T \sigma(t,u)du\right)dt + \sigma(t,T)dW(t) \qquad (4)$$

This is the celebrated Heath-Jarrow-Morton theorem on the term structure of interest rates.

The key input required by the HJM model is the specification of the forward rate volatility function $\sigma(t,T)$. When the volatility function is chosen as a deterministic function of calendar time t and maturity T, it can be shown that the HJM model often leads to closed-form solutions for options on discount bonds. The popular Hull-White model, in fact, is consistent with the following parameterization of the volatility function:

$$\sigma(t,T) = \sigma_0 \exp(-\lambda(T-t))$$

where σ_0 and λ are constants. However, when the volatility function is deterministic (often referred to as *Gaussian* HJM), there's a finite probability that yields can become negative[5]. An alternative specification of the volatility function is to make it dependent on the level of the rate itself, *e.g.*,

$$\sigma(t,T) \Rightarrow \sigma(f(t,T),t,T) = \gamma(t,T)f(t,T) \tag{5}$$

where $\gamma(t,T)$ is a deterministic function of t and T. Although yields are guaranteed to be strictly positive in this case, but as pointed out in the original HJM paper, the *lognornal* condition of (5) would lead to exploding forward rates before T.

The exploding forward rate problem can be avoided if one chooses to work with *discretely* compounded forwards instead of *continously* compounded forward rates that we have been using so far. Also, some recent developments have shown that not only yields can remain stable under the lognormal specification when forward rates are made to be discrete, options on discount bonds can be priced under closed-form formulas as well. This is significant because the parameters of the HJM model can be chosen to match many liquid option prices relatively easily. The following section discusses a version of HJM for discretely compounded forward rates.

Brace-Gatarek-Musiela Model[6]

Recall that we have established the stochastic processes for the (continuously compounded) forward rate and the discount bond price follow:

$$\begin{aligned} dB(t,T) &= r(t)B(t,T)dt - \sigma^*(t,T)B(t,T)dW(t) \\ df(t,T) &= \sigma(t,T)\sigma^*(t,T)dt + \sigma(t,T)dW(t) \end{aligned} \tag{6}$$

where the volatility of discount bond price $\sigma^*(t,T)$ is related to the volatility of (continuously) forward rate $\sigma(t,T)$ through

[5] Market practitioners often dislike negative rates, regardless of how small a probability there is.
[6] For the sake of brevity, some details of the derivation in this section are omitted. See Brace *et al* (1997) for more details.

$$\sigma^*(t,T) = \int_t^T \sigma(t,u)du$$

if we re-parameterize the time indices with current time t and time to maturity $\tau = T - t$, re-write forward rate as $r(t,\tau) = f(t,T) = f(t,t+\tau)$, and bond price as $D(t,\tau) = B(t,T) = B(t,t+\tau)$, then it can be shown that (6) now become

$$dr(t,\tau) = \frac{\partial}{\partial\tau}(r(t,\tau) + \tfrac{1}{2}(\sigma^*(t,\tau))^2)dt + \sigma(t,\tau)dW(t)$$
$$dD(t,\tau) = (r(t,0) - r(t,\tau))D(t,\tau)dt - \sigma^*(t,\tau)D(t,\tau)dW(t) \qquad (6)'$$

define $L(t,\tau)$ as the forward LIBOR rate at time t for a δ -period forward rate maturing at $t + \tau + \delta$ (where $\delta = 0.25$ for US$ LIBOR), and we know that

$$\delta L(t,\tau) = \frac{D(t,\tau)}{D(t,\tau+\delta)} = \exp\left(\int_\tau^{\tau+\delta} r(t,u)du\right) - 1$$

then using (6)' it can be shown that $L(t,\tau)$ follows the SDE

$$dL(t,\tau) = \frac{\partial}{\partial\tau}L(t,\tau)dt + \frac{1}{\delta}(1+\delta L(t,\tau))(\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau))(\sigma^*(t,\tau+\delta)dt + dW(t))$$

if a deterministic function $\gamma(t,\tau)$ exists such that

$$\sigma^*(t,\tau+\delta) - \sigma^*(t,\tau) = \frac{\delta L(t,\tau)\gamma(t,\tau)}{1+\delta L(t,\tau)} \qquad (7)$$

then

$$dL(t,\tau) = \left(\frac{\partial}{\partial\tau}L(t,\tau) + \gamma(t,\tau)L(t,\tau)\sigma^*(t,\tau) + \frac{\delta L^2(t,\tau)\gamma^2(t,\tau)}{1+\delta L(t,\tau)}\right)dt + \gamma(t,\tau)L(t,\tau)dW(t)$$

returning to HJM valuables t and T, set

$$K(t,T) = L(t,T-t)$$

then

$$dK(t,T) = dL(t,T-t) - \frac{\partial}{\partial\tau}L(t,T-t)dt$$
$$= \gamma(t,T-t)K(t,T)\left(\sigma^*(t,T-t)dt + \frac{\delta K(t,T-t)\gamma(t,T-t)}{1+\delta K(t,T)}dt + dW(t)\right)$$

(8)

given $\gamma(t,T-t)$, equation (8) and (7) specify the dynamics followed by the discretely compounded forward rate $K(t,T-t)$, and is known as the *Market Model* of interest rates (or the BGM model).

Brace, Gatarek, and Musiela (1997) showed that under the so-called $T+\delta$ *forward measure*, equation (8) can be simplified to

$$dK(t,T) = \gamma(t,T-t)K(t,T)dW_{T+\delta}(t) \tag{9}$$

where through a *change of measure*, the forward rate process has been made driftless, *i.e.,* a martingle. The change of measure is accomplished by a shift of the Brownian motion

$$W_T(t) = W(t) + \int_0^t \sigma^*(u,T-u)du$$

and a *change of numeraire* from the "rolled-up" money market account to a discount bond. The topics of change of measure and numeraire are beyond the scope of this note[7], but suffice it to say, through some "fancy" stochastic calculus, we could manage to greatly simply the SDE followed by the forward rate to something as simple as (9).

Our new measure, the forward measure, uses the $T+\delta$ maturity discount bond as the numeraire, which means that if the price of an asset $X(t)$ is a martingale under the forward measure, then

$$\frac{X(0)}{P(0,T)} = E^T\left[\frac{X(t)}{P(t,T)}\right]$$

where we used $E^T[]$ to denote the expectation taken under the *T*-maturity forward measure. For an LIBOR caplet settled in arrears, we would then have

$$\frac{C_{T+\delta}(0)}{P(0,T+\delta)} = E^{T+\delta}\left[\frac{(K(T,T)-R_k)^+}{P(T+\delta,T+\delta)}\right] = E^{T+\delta}\left[(K(T,T)-R_k)^+\right]$$

where R_k is the strike rate. From (9) we know $K(T,T)$ is lognormally distributed with time-dependent volatility $\gamma(t,T-t)$, it follows that

$$C_{T+\delta}(0) = \delta P(0,T+\delta)(K(0,T)N(h(0,T)) - R_k N(h(0,T)-\rho(0,T))) \tag{10}$$

where

[7] See Rebonato (1996) for an intuitive discussion of numeraire and how it's related to probability measures.

$$h(0,T) = \frac{1}{\rho(0,T)}\left(\log\frac{K(0,T)}{R_k} + \tfrac{1}{2}\rho^2(0,T)\right) \tag{11}$$

and

$$\rho^2(0,T) = \int_0^T \gamma^2(s,T-s)ds \tag{12}$$

Equation (10) is the familiar *Black Caplet Formula*.

Calibrating HJM/BGM

We have shown that a lognormal version of the HJM model can price the caplet via the Black caplet formula. As we mentioned earlier, the only unknown in the HJM model is the volatility function, which in the case of discretely compounded lognormal model, is connected to the Black caplet volatility via

$$\rho^2(0,T) = \sigma^2_{Black}(T)\cdot T = \int_0^T \gamma^2(s,T-s)ds \tag{12'}$$

where the Black volatility $\sigma_{Black}(T)$ can be obtained directly from market makers of cap and floors, or easily inferred from cap and floor prices. The Black volatility is sometimes referred to as the *terminal* volatility, and the HJM volatility function $\gamma(t,T-t)$ is sometimes called the *instantaneous* volatility.

It's obvious from (12)' that the Black volatility is a time average of the instantaneous volatility function. Given just the average volatility, it's clear that the instantaneous volatility function is not uniquely determined. However, for any given $\gamma(t,T-t)$ that satisfies (12)', caps and floors will be priced exactly. The easiest thing to do is to make $\gamma(t,T-t)$ independent of calendar time *t*, and in this case, we simply have

$$\gamma(t,T-t) = \gamma(T) = \sigma_{Black}(T) \tag{13}$$

Although it's clear that with this choice of the instantaneous volatility, calibration for the HJM model can be easily accomplished, but there're potentially undesirable effects if one choose to ignore calendar time. The most serious being that it would imply a forward rate volatility curve that's shifting backwards as time goes by. For the US$ volatility curve, this would mean, *e.g.*, the *volatility hump* would disappear eventually (in this case after about two years). But in practice as one might notice that the *shape* of the forward rate volatility curve has stayed fairly stable through the years, and it has always displayed a hump at around two years (*i.e.*, $T-t=2$) which would show up in *both* implied volatility curve and volatility curve from analyzing historical data using principle component analysis[8].

A choice of $\gamma(t, T-t)$ that would make the volatility curve stable is to set

$$\gamma(t, T-t) = \gamma(T-t) \tag{14}$$

i.e., the instantaneous/HJM volatility depends only on the *remaining* time to maturity. This is sometimes referred to as the volatility function being *homogenous*. Calibrating the HJM/BGM model using a homogenous volatility function requires more effort for obvious reasons. One typically would have to come up with a parameterized function for $\gamma(T-t)$ that would accommodate the humped shape of the volatility curve and a declining volatility at longer maturities as observed in the market. A potential draw back using a homogeneous volatility function is that it might not fit the Black volatility perfectly if the Black volatility would to decline too rapidly as Rebonato (1996) pointed out. But this is less of a concern in general because (1) a volatility curve rarely undergoes a steep decline at long maturities (it typically rises at short maturities), and (2) one in general does not have to match perfectly the Black volatilities quoted by OTC market makers as those generally carry fairly big "measurement" errors. Since a typical application that would call for a HJM type model is to price exotic derivatives, one in general could live with a good but not perfect calibration of cap/floor prices, and put more emphasis on underlining (rate) distribution assumptions and possibly correlation structures (which we will discuss in the next section).

A third choice would be to chose a HJM volatility function that's "almost" homogeneous, *e.g.*,

$$\gamma(t, T-t) = c(t) g(T-t) \tag{15}$$

where $c(t)$ is only weakly dependent on t and very close to one. This choice of volatility function and other similar variants will fit the Black volatilities exactly in almost all situations, and are sometimes explored by market practitioner. A weakness of this approach is that it has too many independent variables (or degrees of freedom), and the "perfect fitting" parameters sets could be highly unstable.

Multi-Factor HJM

[8] See *e.g.,* Zhu (1999) for an explanation of how to extract volatility curves using principle component analysis.

Although we have so far only discussed the one factor version of the HJM model, one of the strengths of HJM is actually its easy extension to multi-factor. The derivation of multi-factor HJM basically follows the same step as the single-factor case. For an N-factor HJM, the forward rate process would be

$$df(t,T) = \sum_{i=1,N} \sigma_i(t,T) \int_t^T \sigma_i(t,u)du dt + \sum_{i=1,N} \sigma_i(t,T) dW_i(t)$$

where $W_i(t)$, $i = 1,...,N$ are N *independent* Brownian motions. The volatility calibration is straightforward as well. For the two-factor case, *e.g.*, one would have

$$\sigma_{Black}^2(T) \cdot T = \int_0^T (\gamma_1^2(t,T-t) + \gamma_2^2(t,T-t))dt$$

A typical application that would call for a multi-factor HJM is to price a derivative with payoffs dependent on the (imperfect) correlations between forward rates. To keep the number of degrees of freedom manageable, a popular practice is to "factor out" the correlation piece of the volatility functions, and only use one function to account for the over all volatilities. To illustrate this, consider a three-factor HJM with the following volatility functions:

$$\gamma_1(t,T) = g(t,T-t)\phi_1(T-t)$$
$$\gamma_2(t,T) = g(t,T-t)\phi_2(T-t)$$
$$\gamma_3(t,T) = g(t,T-t)\phi_3(T-t)$$

where $g(t,T)$ would satisfy

$$\sigma_{Black}^2(T) \cdot T = \int_0^T g^2(t,T-t)dt$$

and $\phi_{1,2,3}(T-t)$ are constrained by

$$\phi_1^2(T-t) + \phi_2^2(T-t) + \phi_3^2(T-t) = 1 \qquad (16)$$

Its easy to show that the correlation between forward rates $f(t,T_1)$ and $f(t,T_2)$ is given by

$$\phi_1(T_1-t)\phi_1(T_2-t) + \phi_2(T_1-t)\phi_2(T_2-t) + \phi_3(T_1-t)\phi_3(T_2-t) \qquad (17)$$

i.e., independent of $g(t,T-t)$. Here we have implicitly assumed that correlations are time-homogeneous.

A set of correlations (perhaps from historical data) can be used together with (17) to determine our ϕ functions. The market practitioners typically choose the functional form of ϕ's from a set of elementary functions. An extremely robust set would be the trigonometry functions. In the three-factor case, we would have

$$\begin{aligned} \phi_1(\tau) &= \cos(\alpha(\tau)) \\ \phi_2(\tau) &= \sin(\alpha(\tau))\cos(\beta(\tau)) \\ \phi_3(\tau) &= \sin(\alpha(\tau))\sin(\beta(\tau)) \end{aligned} \tag{18}$$

where $\alpha(\tau)$ and $\beta(\tau)$ are functions of $\tau = T - t$, and can be to be fitted using (17) to the correlation data. The nice thing about (18) is that it satisfies the constraint (16) for any choice of angle functions $\alpha(\tau)$ and $\beta(\tau)$.

Principle Component Analysis (PCA) was used to obtain the actual functional form of the factors. Appendix A at the end of this document provides and introduction of PCA, and an actual example of how PCA is performed on the yield curve.

Actual analysis of the Principle Components of the forward curve can be complicated by the type of yield curve smoothing method one chose to use. Figure-1 shows a typical result[9] from a fit to forward rate correlation data. We see that ϕ_1 can be interpreted as contributing to a parallel movement of the forward curve, ϕ_2 is responsible for a twist of the yield curve, and ϕ_3 corresponds to a "bending" mode.

[9] Using swap rates from 1998-2000. Actual analysis of the Principle Components of the forward curve can be complicated by the type of yield curve smoothing method one chose to use. For the results shown in Figure-1, we use a three-factor empirical term structure model to generate daily yield curves and perform PCA on the forward curves. Alternatively, one can also perform principle component analysis on the monthly change of the forward curve (using any smoothing method), but that usually would require more data.

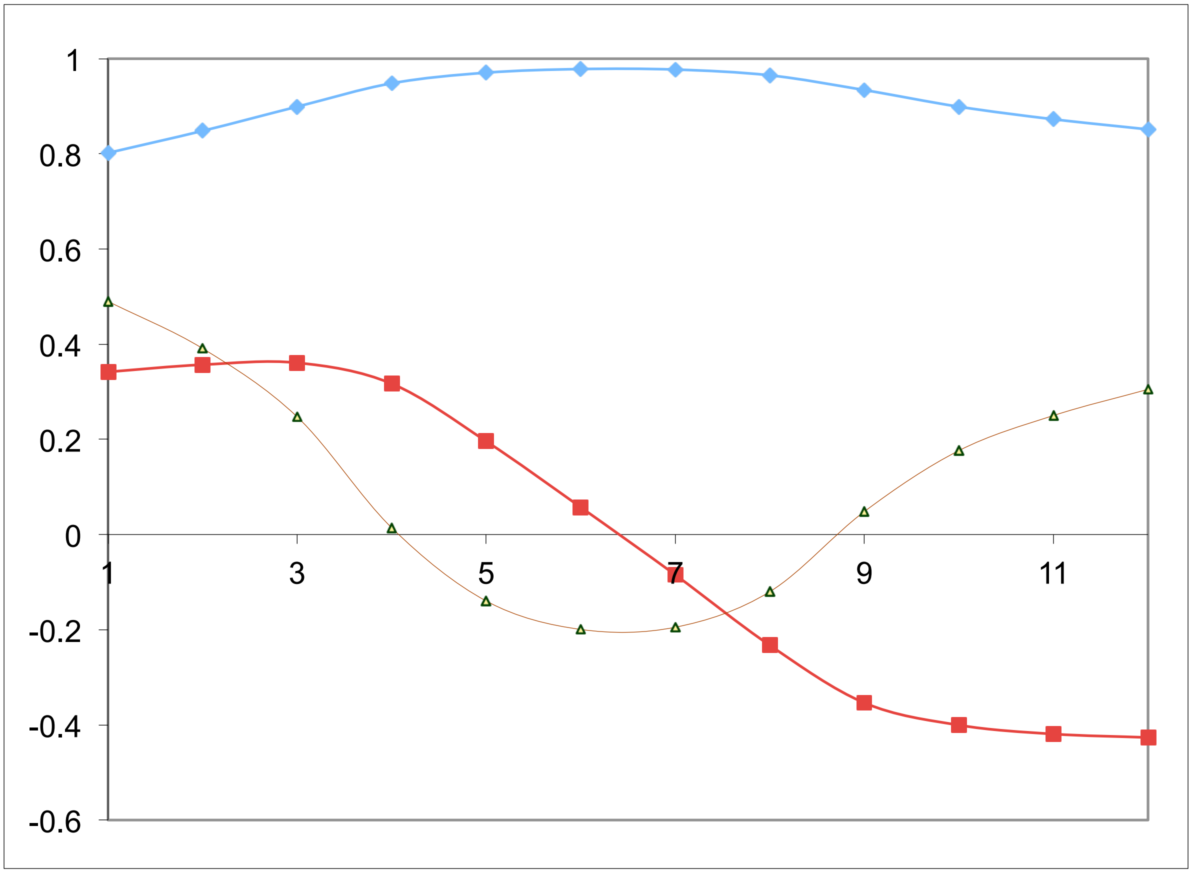

Figure-1 ϕ_1 (top), ϕ_2 (with squares), and ϕ_3 (with diamonds) from fitting forward rate correlations

Model Implementation Details

We follow the steps outlined in John Hull[10] for our model implementation at Portfolio Management. Define:

$F_k(t)$: Forward rate between t_k and t_{k+1} as seen at time *t*. expressed with a compounding period of $\delta = t_{k+1} - t_k = 0.25$

$m(t)$: Index for the next reset date at time *t*; this means that $m(t)$ is the smallest integer such that $t \leq t_{m(t)}$

$\varsigma_{k,q}(t)$: Volatility of $F_k(t)$ at time *t* for the *q*th factor

[10] John Hull and Alan White, *Forward Rate Volatilities, Swap Rate Volatilities and the Implementation of the Libor Market Model.* August, 1999. http://www.rotman.utoronto.ca/~amackay/fin/libormktmodel2.pdf

If forward rates are assumed to be lognormal, the discretization equation for the forward rates is:

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta F_i(t) \sum_{q=1}^{p} \varsigma_{i,q}(t)\varsigma_{k,q}(t)}{1 + \delta F_i(t)} dt + \sum_{q=1}^{p} \varsigma_{k,q}(t) dz_q \qquad (19)$$

For a normal model, we will have:

$$dF_k(t) = \sum_{i=m(t)}^{k} \frac{\delta \sum_{q=1}^{p} \varsigma_{i,q}(t)\varsigma_{k,q}(t)}{1 + \delta F_i(t)} dt + \sum_{q=1}^{p} \varsigma_{k,q}(t) dz_q \qquad (20)$$

Current production version of the model used for daily MSR valuation at Portfolio Management is a 3-factor normal implementation of BGM model[11]. The factor loadings for the 3-factors follow equation (18) and Figure-1 described in the previous section, and the volatilities are assumed to be a non-homogenous form:

$$\varsigma_{k,1}(t) = \phi_{1,k}(t)\Lambda_k$$
$$\varsigma_{k,2}(t) = \phi_{2,k}(t)\Lambda_k$$
$$\varsigma_{k,3}(t) = \phi_{3,k}(t)\Lambda_k$$

ie, Λ_k depends only on the forward rate index, but not calendar time *t*. The rationale for picking a non-homogenous form of the volatility function is its ease of calibration (as show in equation (13)) and it is consistent with how market participants' view of the evolution of volatilities. When calibrated to cap vols, Λ_k are simply the caplet vols themselves[12].

Alternatively, one can also calibrate to a swaption vol series or any swaption vol grid. Although no closed form formula exists for the value of a swaption under the BGM model, extremely accurate approximations can still be obtained. Following that of John Hull, the swaption volatility can be approximated as:

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[11] In valuing the MSR asset, since both prepayments and cash flows for the servicing will depend on the long and short end of the yield curve, we believe a multi-factor interest model is better suited for modeling the future cash flows of the asset.

[12] This is part of the attractiveness of the BGM model since in this case one can use both the yield curve and caplet vol curve as inputs to the model and therefore no iterative calibration procedures are needed.

$$\sqrt{\frac{1}{t_n}\sum_{j=0}^{n-1}\left\{\delta\sum_{q=1}^{3}\left[\sum_{k=n}^{N}\frac{\delta\varsigma_{k,q}F_k(0)\gamma_k(0)}{1+\delta F_k(0)}\right]^2\right\}} \tag{21}$$

Where the relevant swaption is an option on a swap lasting from t_n to t_{N+1} with reset dates at times $t_n, t_{n+1}, \ldots, t_N$, and,

$$\gamma_k(t) = \frac{\prod_{j=n}^{N}[1+\delta F_j(t)]}{\prod_{j=n}^{N}[1+\delta F_j(t)]-1} - \frac{\sum_{i=n}^{k-1}\delta\prod_{j=i}^{N}[1+\delta F_j(t)]}{\sum_{i=n}^{N}\delta\prod_{j=i+1}^{N}[1+\delta F_j(t)]} \tag{22}$$

Equation-21,22 basically relate the short rate volatilities to that of long rate volatilities using the shape of the yield curve. By the complexity of the relationship, one would need some kind of iterative procedure when calibrating the short rate volatilities given the long rate volatilities.

Current production implementation of the BGM model in Portfolio Management calibrates to the series of x into 10-yr swaptions. We started out assuming a piecewise linear function for the short rate vol curve with the same number of knot points as the swaption series. These knot points are solved using a non-linear least square method by minimizing the error between the calculated swaption vol (equation-21) and that of the market. To ensure smoothness of the piecewise linear vol curve, the objective function in least square fitting includes penalty terms to make the change of knot points in the piecewise curve less "jumpy":

$$\sum_{i=1}^{N}[\sigma_i^{model} - \sigma_i^{market}]^2 + \sum_{i=1}^{N} w_i(p_i - p_{i-1})^2$$

where p_i are the knot pints for the piecewise linear curve, and the weights w can be obtained empirically to constrain smoothness[13].

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[13] For example, the robustness of the weights *w* can be empirically tested against historical market data. Since our swaption vols are closed form, this can be performed relatively easily.

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Appendix A

Principle Components Analysis of Risk Factors

When dealing with multiple, but *correlated* risk factors, the Principle Components Analysis (PCA) technique can be used to transform the original risk factors into a new set of risk factors that are *independent* of each other and can be ranked by their relative importance (hence the meaning of "principle components"). In this note, we give a simple and intuitive explanation of how this can be accomplished.

Let's consider the case of two risk factors. The dynamics of their statistical processes can be described by the following simple equations:

$$\begin{aligned} dX_1(t) &= \mu_1 dt + \sigma_1 dW_1(t) \\ dX_2(t) &= \mu_2 dt + \sigma_2 dW_2(t) \end{aligned} \qquad (1)$$

where μ_1, μ_2 are the drift rates and σ_1, σ_2 are the standard deviations for our two risk factors X_1, X_2. dW_1 and dW_2 are correlated random (gaussian) deviations with correlation coefficient ρ and variance dt. In practice, one is most interested in the standard deviations σ_1 and σ_2. These can be estimated from the time series of X_1, X_2. For example, by taking the expected value of $(dX_1)^2$, we would have

$$\begin{aligned} E\left[(dX_1)^2\right] &= E\left[(\mu_1 dt + \sigma_1 dW_1)\cdot(\mu_1 dt + \sigma_1 dW_1)\right] \\ &= E\left[\mu_1^2 (dt)^2 + 2\mu_1\sigma_1 dt dW_1 + \sigma_1^2 (dW_1)^2\right] \end{aligned}$$

from gaussian statistics, we know $dW_1 \cong \sqrt{dt}$, $(dW_1)^2 \cong dt$, and therefore for small dt, the first two terms in the above equation can be dropped, and we will have

$$E\left[(dX_1)^2\right] \cong E\left[\sigma_1^2 dt\right] = \sigma_1^2 dt$$

This also means that for the purposes of studying volatilities, one can basically ignore the drift μ_1, or in other words, set $\mu_1 = 0$. Our simplified version of equation (1) can be written in vector form as

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 dW_1 \\ \sigma_2 dW_2 \end{pmatrix} \qquad (2)$$

What we wish to do is instead of working with correlated random (gaussian) variables W_1 and W_2, we find two new random variables $\widetilde{W}_1$ and $\widetilde{W}_2$, and specify equation (2) in terms of the new random variables. A simple way to express this to write (2) as

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 dW_1 \\ \sigma_2 dW_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} d\widetilde{W}_1 + \sigma_{12} d\widetilde{W}_2 \\ \sigma_{21} d\widetilde{W}_1 + \sigma_{22} d\widetilde{W}_2 \end{pmatrix} \quad (3)$$

rearranging it slightly, we could also write (3) as

$$\begin{pmatrix} \sigma_1 dW_1 \\ \sigma_2 dW_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} s_1 d\widetilde{W}_1 \\ s_2 d\widetilde{W}_2 \end{pmatrix} \quad (4)$$

our objective now is to find (s_1, s_2) and matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Multiplying the column vectors in (4) by their corresponding row vectors, we have

$$\begin{pmatrix} \sigma_1 dW_1 \\ \sigma_2 dW_2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 dW_1 & \sigma_2 dW_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} s_1 d\widetilde{W}_1 \\ s_2 d\widetilde{W}_2 \end{pmatrix} \begin{pmatrix} s_1 d\widetilde{W}_1 & s_2 d\widetilde{W}_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^T$$

which is

$$\begin{pmatrix} \sigma_1^{\,2} (dW_1)^2 & \sigma_1 \sigma_2 dW_1 dW_2 \\ \sigma_1 \sigma_2 dW_1 dW_2 & \sigma_2 (dW_2)^2 \end{pmatrix} = A \cdot \begin{pmatrix} s_1^{\,2} (d\widetilde{W}_1)^2 & s_1 s_2 d\widetilde{W}_1 d\widetilde{W}_2 \\ s_1 s_2 d\widetilde{W}_1 d\widetilde{W}_2 & s_2^{\,2} (d\widetilde{W}_2)^2 \end{pmatrix} \cdot A^T$$

Taking the expectations (average) on both sides, using $E\left[(dW_{1,2})^2\right] = E\left[(d\widetilde{W}_{1.2})^2\right] = dt$, $E\left[dW_1 dW_2\right] = \rho dt$, and $E\left[d\widetilde{W}_1 d\widetilde{W}_2\right] = 0$, we have

$$\begin{pmatrix} \sigma_1^{\,2} dt & \sigma_1 \sigma_2 \rho dt \\ \sigma_1 \sigma_2 \rho dt & \sigma_2^{\,2} dt \end{pmatrix} = A \cdot \begin{pmatrix} s_1^{\,2} dt & 0 \\ 0 & s_2^{\,2} dt \end{pmatrix} \cdot A^T \quad (5)$$

dropping out dt, we can rewrite the above simply as

$$\Sigma = ASA^T \quad (5)'$$

where Σ is the familiar covariance matrix, and S is a diagonal matrix. From linear algebra, we know equation (5) can be solved as a common eigenvalue problem, or sometimes referred to as the diagonalization of a matrix[14]. Since the covariance matrix is symmetric, we know that A is orthogonal, *i.e.*, $A^{-1} = A^T$. (5)' can also be written as

[14]Numerous algorithms for solving the eigenvalue problem can be found in Press *et al* (1992), the most famous one being the Jacobi method.

$$S = A^T \Sigma A \qquad (6)$$

The eigenvalues make up the diagonal matrix S, and it's not difficult to see that equation (6) also holds for more than two risk factors, *i.e.,* covariance matrix of any size. We interpret the largest eigenvalues and its corresponding eigenvector (which is the corresponding column vector in matrix A) for the covariance matrix as the 1st *principle component*. Similarly, one would also have 2nd principle component, 3rd, etc for other eigenvalues arranged in descending order. The *weight* of principle component i is defined as the weight of eigenvalue s_i^2 among all eigenvalues:

$$w_i = \frac{s_i^2}{\sum_j s_j^2}$$

One notices that so far we have expressed our eigenvalues as the square of a real number. This is justified since a well behaved covariance matrix is positive definite[15], therefore as we know from linear algebra that all eigenvalues of a positive definite matrix are strictly positive[16].

It's time to look at a simple example. Suppose we have

$$\Sigma = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}$$

Diagonalizing this matrix gives

$$S = \begin{pmatrix} 1.8 & 0 \\ 0 & 0.2 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

This implies that the weight of the 1st principle component is 0.9 (or 90%), and equation (3) would be

[15] Among other properties, positive definite would guarantee, for example, the absolute value of off-diagonal elements of the covariance matrix is not too large as to imply larger than one absolute correlation coefficients.

[16] When diagonalizing a covariance matrix of high dimension using certain numerical algorithms, depending on the precision of the iterative procedure, it's sometimes possible to get very small negative eigenvalues. In that case, it's always safe to assume that those eigenvalues/eigenvectors do not have much weight and can be ignored.

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix} = \begin{pmatrix} 0.949 \cdot d\widetilde{W}_1 + 0.316 \cdot d\widetilde{W}_2 \\ 0.949 \cdot d\widetilde{W}_1 - 0.316 \cdot d\widetilde{W}_2 \end{pmatrix}$$

The 1st principle component in this case can also be interpreted as the (independent) random factor $\widetilde{W}_1$, and one should expect it to account for 90% of the variance.

PCA of Treasury Rates

Next we look at a more realistic example by analyzing the covariance matrix of Treasury rates.

The most actively traded Treasuries are the seven newly issued "on-the-run" Treasuries which consist: one-, three-, and twelve-month bills, two- and five-year notes, and 10- and 30-year bonds[17]. Although at any given day, there are no exact two-, five-, 10-year, etc Treasuries trading except the days when US Treasury Department are auctioning off new Treasuries, most people are comfortable working with the so-called "generic" Treasury rates. These are the *yield-to-maturities* (yields in short from now on) of exact two-, five-, 10-year, etc. bonds carefully extrapolated from actual on-the-run yields and published daily after market close by the Federal Reserve Bank of New York. These yields are often referred to as the CMT (Constant Maturity Treasury) yields, and they are more useful than the yields of real on-the-run bonds since their changes can be compared "fairly" on a day to day basis. This is because an actual bond would "age", and its yield next day would be a yield of a different maturity[18].

Table-1 lists the annualized percentage volatilities of the yields of seven Treasuries for 260 trading days prior to October 1998, and Table-2 is the correlation matrix. Here we prefer to list the correlation matrix instead of the covariance matrix since the correlation matrix can be thought of as a "normalized" covariance matrix, and therefore its easy to visualize the covaraince structure. Table-3 lists the factor weights for the principle components in descending order. As we see, the first three principle components accounts for about 95% of the total, or in other words, the first three principle components can explain 95% of the total variations of the Treasury yield curve.

Figure-1 plots the *volatility curves*, or sometimes referred as the *factor loadings* for the first three principle components (factors) for the Treasury yields. The first factor is responsible for the parallel shift of the yield curve, the second factor would produce a twist of the yield curve with yields of short and long maturities moving in the opposite direction, and the third factor roughly contributes to sort of a bending of the yield curve.

[17] For a while, there were eight "on-the-run" Treasuries, but US Treasury Department stopped issuing the three-year note since May 1998.
[18] In principle, comparing the yield of a bond or note to that of a bill is not strictly "fair" because the yield of a bill is a discount yield while a bond yield is sort of an "average" yield with semi-annual compounding (see, *e.g.,* John Hull (1997)). But for the purposes of studying the volatilities of yields, we will ignore this difference.

We see using the principle components analysis on the yields of the Treasuries, the movements of the whole yield curve can be reasonably described by three risk factors, instead of seven. The stochastic differential equations (SDE) governing the dynamics of the yield curve can be simplified from:

$$dR_i(t) = \mu_i dt + \sigma_i dW_i \qquad i = 1,2,...,7$$

with seven correlated random factors $W_{i=1,2,...,7}$, and 21 correlation coefficients (not shown above) to a simpler set of SDEs:

$$dR_i(t) = \mu_i dt + \sum_{j=1}^{3} s_{i,j} d\widetilde{W}_j(t) \qquad i = 1,2,...,7$$

with three uncorrelated random factors $\widetilde{W}_{i=1,2,3}$.

Maturities	*Volatilities*
0.25	0.227684
0.50	0.185491
1.0	0.189358
2.0	0.229206
5.0	0.232558
10.0	0.214586
30.0	0.148952

Table-1 Volatilities of Treasury rates (October 1997- October 1998)

	0.25	0.50	1.0	2.0	5.0	10.0	30.0
0.25	1	0.609693	0.601088	0.429643	0.331705	0.280194	0.274538
0.50	0.609693	1	0.791064	0.620346	0.484432	0.391611	0.309078
1.0	0.601088	0.791064	1	0.851136	0.754773	0.675219	0.564977
2.0	0.429643	0.620346	0.851136	1	0.925525	0.849766	0.735161
5.0	0.331705	0.484432	0.754773	0.925525	1	0.959996	0.847104
10.0	0.280194	0.391611	0.675219	0.849766	0.959996	1	0.911944
30.0	0.274538	0.309078	0.564977	0.735161	0.847104	0.911944	1

Table-2 Correlation matrix of Treasury rates (October 1997- October 1998)

	Weights
1	0.706017
2	0.182824
3	0.062619
4	0.023265
5	0.013225

6	0.008928
7	0.003124

Table-3 Factor weights for the principle components

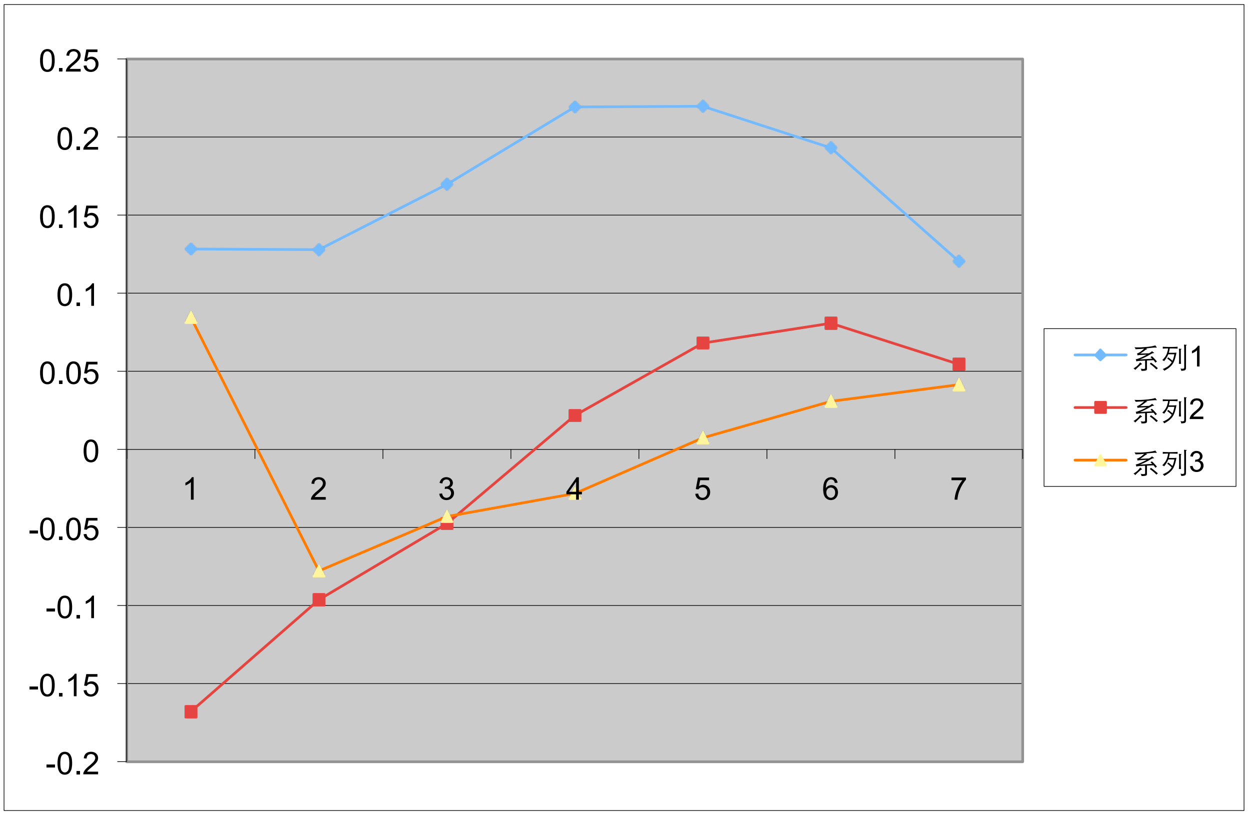

Figure-1 Yield Factors

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Appendix B

Maximum Smooth Forward Curves

1. Introduction

Building a zero-coupon yield curve from a finite number of coupon-bearing bonds has been a basic technique in finance for a long time. However, recent developments of the term structure model of interest rates and the proliferation of increasing complex derivative products call for "well-behaved " or "smooth" yield curves. On the other hand, how to quantitatively define "well-behaved" or "smooth" yield curves, and further to build it is not entirely a trivial matter.

Fundamentally, a yield curve that is extracted from only a finite number of bonds is not unique. Different methods or more specifically, the different assumptions of parametric forms result in different yield curves. The most popular method for building the yield curve has been the cubic spline interpolation method. Mathematically, the cubic spline interpolation formula is a function that fits a given set of data and is continuous through the second derivative, both within a data interval and at its boundaries. The application of the cubic spline method to fit a yield curve to the prices of U.S.Treasury Securities was first introduced by McCulloch (1975). It was later proved that there is no smoother function, of any functional form, that fits the observable data points and is continuous and twice differentiable at the knot points than a cubic spline. Therefore, if the objective of the analyst is to get the smoothest possible yield curve, then the cubic spline of yield produces the smoothest possible yield curve. If the objective of the analyst is the smoothest possible discount bond price function, then a cubic spline of zero coupon bond prices produces the smoothest price curve. However, the forward rate curve derived from the cubic spline approach of yields is not twice differentiable at the knot points, and hence, it is not "smooth" enough from the point of view of forward rates. The lack of higher order derivatives of the yield curve is undesirable for most interest rate models.

Recently, Adams and van Deventer defined a criterion for the best fitting yield curve to be "Maximum Smooth" for the forward rates. They also introduced a mathematical measure of smoothness as an objective criterion for choosing the yield curve smoothing methods. According to Adam and van Deventer, the "smoothness" can be mathematically defined as the value Z given by the formula

$$Z = \int_0^T ds[f''(s)]^2$$

where the function $f(s)$ is a function used to fit the observed data. Based on this criterion, i.e., the "Maximum Smoothness" for the forward rate, Adam and van Deventer developed a powerful technique to fit yield curves. This technique can be used for fitting the yield curve with one explicit function that is both consistent with all the observed points on the yield curve and provides the smoothest possible forward rate curve.

In this note, we are going to compare yield curves constructed from methods extended from cubic spline and the maximum smoothness approaches. The data we are going to fit are the on-the-run U.S. Treasury Securities: Treasury bills with maturities of 3-month, 6-month, and 1-year, Treasury note with maturities of 2-year, and 5-year, and Treasury bonds with maturities of 10-year and 30-year. In section 2, we are going to briefly present the algorithm of the cubic spline and various boundary conditions. We will discuss and compare the differences of zero-coupon yield curve and forward rate curve constructed from applying cubic spline to zero yields and applying cubic spline to zero-coupon prices respectively. The results from different boundary conditions are also compared. Our results show that yield curve and forward curve constructed from cubic spline of zero-coupon prices are less oscillating than those constructed from cubic spline of yields. In section 3, we present the algorithm of maximum smoothness of forward rates. The yield curve, and forward rate curve built from the maximum smoothness method are compared to those built from the cubic spline of yields and cubic spline of price srespectively. We conclude that yield curve built from the maximum smoothness of forward rates method has much less oscillation in the yield curve than applying cubic spline to yields directly.

2. Cubic Spline Method

$$y_i = f(x_i)$$

Assume that we are given a set of data

where *i = 1, ..., N.* The set of data can be the yields of zero coupon bonds $y_1, y_2, \ldots, y_N$ at the maturities $x_1, x_2, \ldots, x_N$, or the price of zero coupon bonds $P_1, P_2, \ldots, P_N$ at the maturities $x_1, x_2, \ldots, x_N$.

The objective of cubic spline is to find a function that fits the set of data , and is twice differentiable and continuous in the first and second derivatives within all the intervals and boundaries.

Without losing generality, a cubic formula can be written as

$$y = Ay_j + By_{j+1} + Cy''_j + Dy''_{j+1}$$

where y''_j is the second derivative of $f(x)$ at knot x_j **,** and A , *B*, *C*, and *D* are defined as:

$$A = \frac{x_{j+1} - x}{x_{j+1} - x_j} \qquad B = \frac{x - x_j}{x_{j+1} - x_j}$$

$$C = \frac{1}{6}(A^3 - A)(x_{j+1} - x_j)^2 \qquad D = \frac{1}{6}(B^3 - B)(x_{j+1} - x_j)^2$$

With a formula written in this form, the requirement that function $f(x)$ and its second order derivative is continuous at knots are satisfied. The restrain that its first derivative is continuous at knots leads to the equations:

$$\frac{x_j - x_{j-1}}{x_{j+1} - x_{j-1}} y''_{j-1} + 2y''_j + \frac{x_{j+1} - x_j}{x_{j+1} - x_{j-1}} y''_{j+1} = \frac{6}{x_{j+1} - x_{j-1}} \left\{ \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}} \right\}$$

where $j = 2, ..., N-1$.

These are N-2 linear equations in the N unknown variables $y''_j, j = 1, ..., N$. Thus we need two more equations to solve the unknowns. The additional two equations can be obtained by boundary conditions at two ends.

- At the low end of boundary, a *natural* spline boundary condition is imposed $y''_1 = 0$.
- At the up end, we can impose a natural cubic spline boundary condition: $y''_N = 0$, or a flat yield boundary condition: $y'_N = 0$ for cubic spline of yields

  or a flat yield boundary condition: $y''_{N-1} + 2y''_N = \frac{6}{x_N - x_{N-1}} \left\{ \frac{y_N \ln(y_N)}{x_N} - \frac{y_N - y_{N-1}}{x_N - x_{N-1}} \right\}$

  for cubic spline of prices.

With these two additional boundary conditions, we can solve the *N* unknown variables and obtained the yield curve that fit the observed market data.

To compare yield curves built from cubic spline of yields and cubic spline of prices, we have built yield curves using the on-the-run U.S. Treasury Securities data provided by Bloomberg. Table 1 lists the seven on-the-run Treasury Securities traded on March 23, 1999.

Among the seven types of securities, the last four are coupon-bearing bonds. The cubic spline method described so far can only be applied directly to zero-coupon bonds. For a coupon-bearing bond, an iteration algorithm must be incorporated in order to apply the cubic spline method. The detailed implementation of this iteration algorithm has been described and discussed in Zhu (1999), and in van Deventer and Kenji Imai (1996). An alternative way to solve the yield curve of coupon-bearing bonds is to solve the N+1 nonlinear equations as mentioned in Yekutieli (1999). Here we employ the iteration method.

Table –1 Treasury Data on March 22, 1999

ID	Name	Maturity	**Coupon**	Market Quotes	Present Value
5100	GB3	06/17/1999	0	4.395	98.9501
5101	GB6	09/16/1999	0	4.445	97.8145
5102	GB1	03/02/2000	0	4.515	95.6731
5103	GT2	02/28/2001	5	99.8984	100.211
5105	GT5	02/15/2004	4.75	98.4297	98.9021
5106	GT10	11/15/2008	4.75	96.6094	98.2889
5107	GT30	02/15/2029	5.25	95.4375	95.9596

Figure 1 shows the yield curves from the cubic spline of yields and the cubic spline of prices. In the both methods, the flat yield boundary condition are used. Both yield curves have oscillations around 5-year maturity. However, the oscillation in the yield curve from the cubic spline of prices is much milder than that from the cubic spline of yields. Figure 2 shows the forward rate curves from cubic spline of yields and cubic spline of prices. Both forward rate curves show wild fluctuations for the maturity ranging from zero to 10 years. The results illustrate that forward rates derived from cubic spline method are very volatile, in addition to the fact that forward rates from the cubic spline method are not twice differentiable.

Figure-1 Zero-coupon yield of U.S. Treasury on March 20, 1990 built from different methods: yd(4) is the yield curve built from the cubic spline of price, yd(5) is the yield curve is built from the cubic spline of yield, and yd(knots) is zero-yield at maturities.

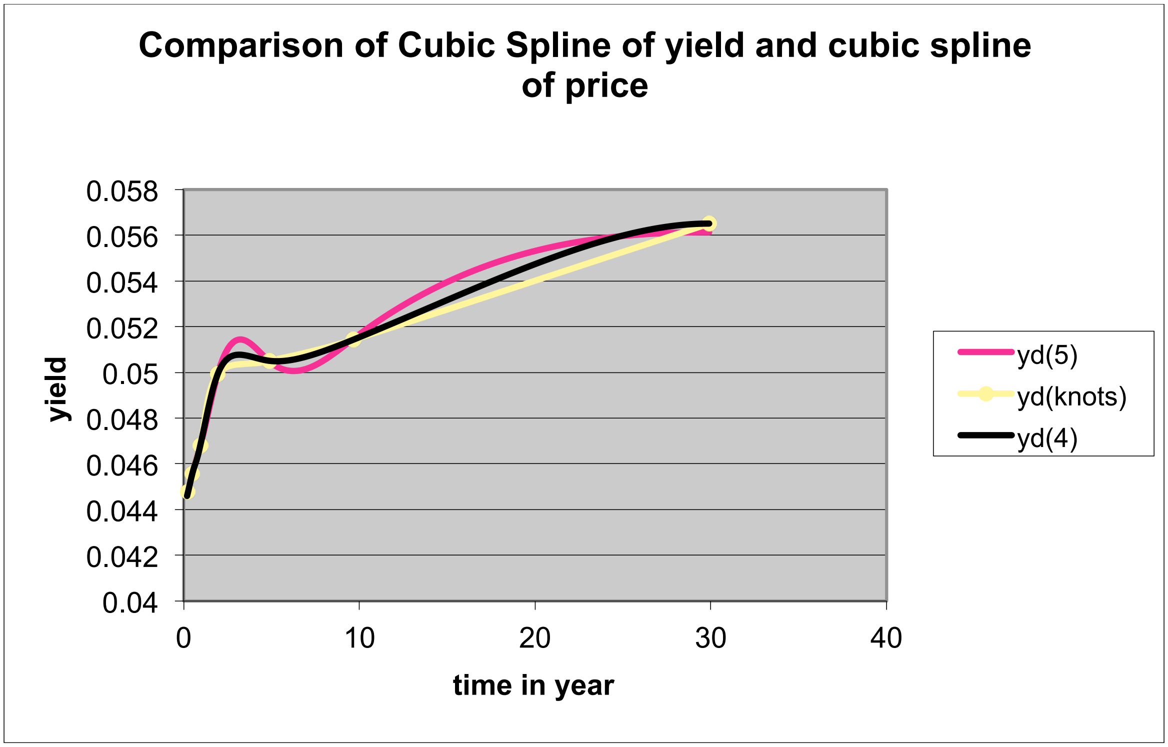
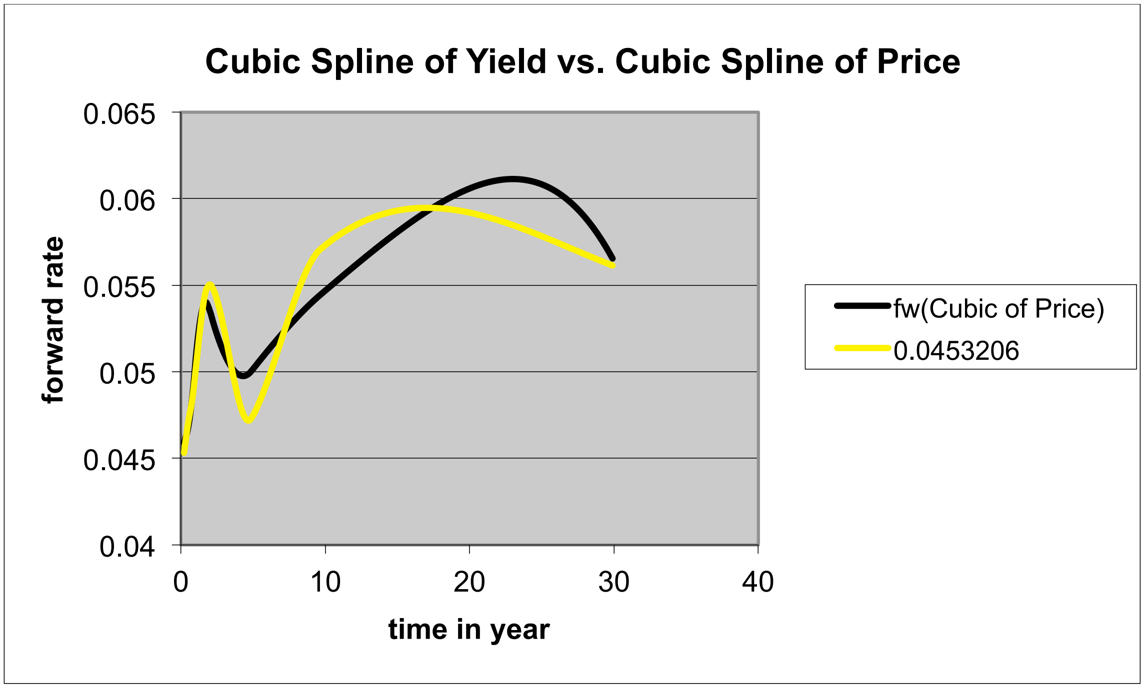


Figure-2 Forward rate curves of U.S. Treasury on March 20, 1990 built from different methods, the dark curve is built from the cubic spline of price, the light curve is built from the cubic spline of yield.



2. Maximum Smoothing Method

In order to remedy the problems of cubic spline method, Adam and van Deventer came up with the Maximum Smoothing method to obtain a "well-behaved", smoother forward rate curve. The forward rate curve obtained by Maximum Smoothing method has the following feature:

- The forward rate curve is continuous and twice differentiable.
- The forward rate is the smoothest curve of any of the family of curve that are continuous , twice differentiable, and consistent with the observed data.

It can be shown that the smoothest possible forward rate curve satisfying the above requirements consists of a *quartic* forward rate function that fits between each knot point (van Deventer, 1996)

Suppose that we are given a set of prices of zero-coupon bonds $P_1, P_2, \ldots, P_N$ with maturities $t_1, t_2, \ldots, t_N$. The Maximum Smoothing forward rate curve is given by
$f(t) = e_i t^4 + d_i t^3 + c_i t^2 + b_i t + a_i$, for $t_{i-1} < t \le t_i$, $i = 1, 2, \ldots, N+1$
where $0 = t_0 < t_1 < t_2 < \ldots < t_N < t_{N+1} = T$.

The coefficients a_i, b_i, c_i, d_i, and e_i satisfy the following equations

$$e_{i+1} t_i^4 + d_{i+1} t_i^3 + c_{i+1} t_i^2 + b_{i+1} t_i + a_{i+1} = e_i t_i^4 + d_i t_i^3 + c_i t_i^2 + b_i t_i + a_i$$

$$4e_i t_i^3 + 3d_i t_i^2 + 2c_i t_i + b_i = 4e_{i+1} t_i^3 + 3d_{i+1} t_i^2 + 2c_{i+1} t_i + b_{i+1}$$
$$12e_i t_i^2 + 6d_i t_i + 2c_i = 12e_{i+1} t_i^2 + 6d_{i+1} t_i + 2c_{i+1}$$
$$24e_i t_i + 6d_i = 24e_{i+1} t_i + 6d_{i+1}$$
$$\frac{1}{5}e_i(t_i^5 - t_{i-1}^5) + \frac{1}{4}d_i(t_i^4 - t^4{}_{i-1}) + \frac{1}{3}c_i(t_i^3 - t_{i-1}^3) + \frac{1}{2}b_i(t_i^2 - t_{i-1}^2) + a_i(t_i - t_{i-1}) = -\log\left(\frac{P_i}{P_{i-1}}\right)$$
where $P_0 = 1$, and $i = 1, 2, ..., N$

Thus, we have 5N equations in 5N+5 unknown variables. The additional five equations can be obtained by boundary conditions.

- At the low end of boundary, the forward rate curve is instantaneously straight, so that $c_1 = 0$, $d_1 = 0$, and $c_1 = 0$.
- At the up end of the boudary, we can either have straight forward rate boundary condition $f''(t_N) = 2c_N + 6d_N t_N + 12e_N t_N^2 = 0$,
  or flat yield boundary condition $y'(t_N) = -\frac{1}{t_N{}^2}\int_0^{t_N} ds f(s) + \frac{1}{t_N} f(t) = 0$.
  The flat yield boundary condition can be further written as
  $$\sum_{i=1}^{N}\left\{\frac{1}{5}e_i(t_i^5 - t_{i-1}^5) + \frac{1}{4}d_i(t_i^4 - t_{i-1}^4) + \frac{1}{3}c_i(t_i^3 - t_{i-1}^3) + \frac{1}{2}b_i(t_i^2 - t_{i-1}^2) + a_i(t_i - t_{i-1})\right\}$$
  $$-\left(e_N t_N^4 + d_N t_N^3 + c_N t_N^2 + b_N t_N + a_N\right) = 0$$

With these five additional equations of boundary conditions, we can solve the 5*N+5* unknown variables and obtained the forward rate curve that fits the data of zero-coupon bonds. To apply the Maximum Smoothing method to coupon-bearing bonds, we again use the iteration method previously mentioned.

Figure 3 shows the yield curve built from the U. S. Treasury data in Table-1 by using the Maximum Smoothing method. To compare the Maximum Smoothing method to the cubic spline of yields, and the cubic spline of prices, we have also plotted the yield curves by the last two methods. Our results demonstrate that yield curve constructed from the maximum smoothing method is much smoother than that constructed from the cubic spline of the yields. On the other hand, it is surprising that the yield curve built from the maximum smoothing method is very close to that built from the cubic spline of prices at time range $0 < t < 11$ years. Figure 4 shows the forward rate built from the maximum smoothing method and those built from the cubic spline of the yields and cubic spline of prices. It is obvious that the forward rate curve from the maximum smoothing method is smoother that those from the last two methods.

Figure-3 Zero-coupon yield curves of U.S. Treasury on March 23, 1999 built from different methods. yd(7):maximum smoothness; yd(4): cubic spline of price; yd(5) :cubic spline of yield; yd(knots): zero yield at maturities of bonds.

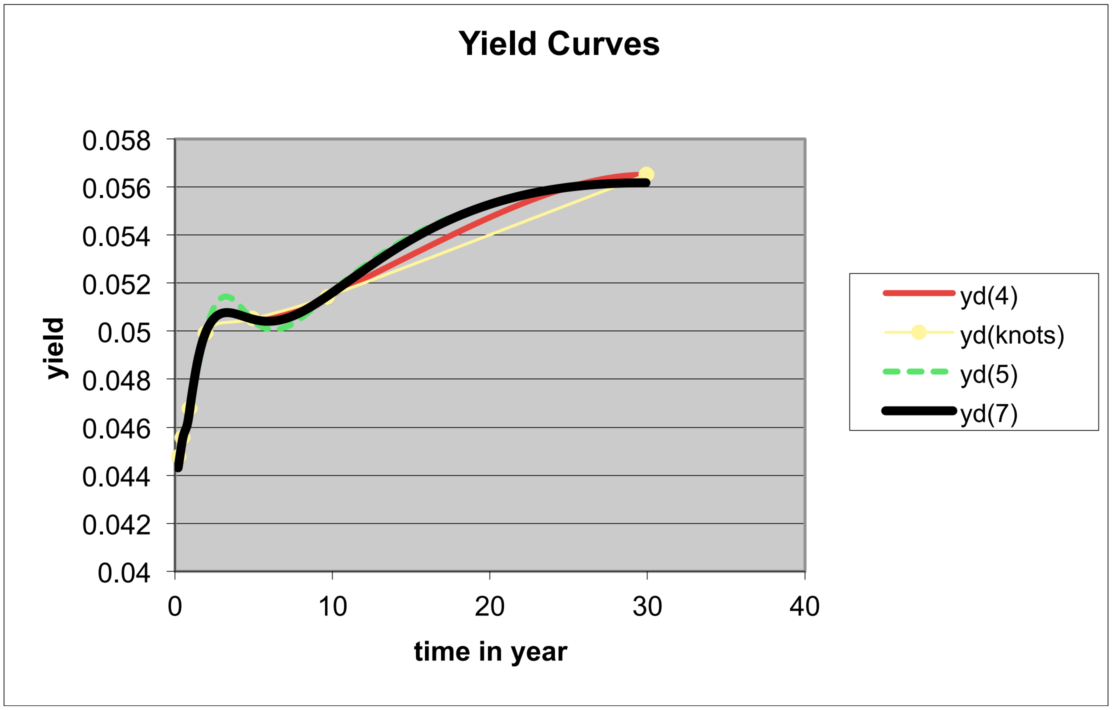
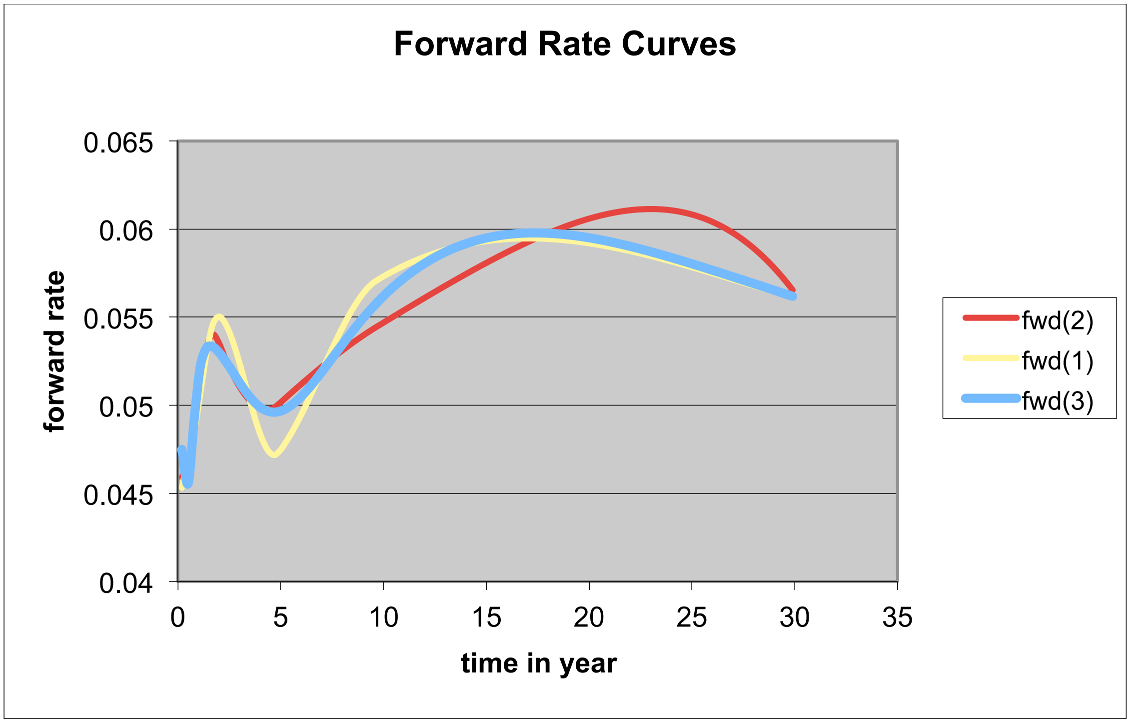


Figure-4 Forward rate curves of U.S. Treasury on March 23, 1999 built from different methods. fwd(3): maximum smoothness; fwd(2):cubic spline of price; fwd(1):cubic spline of yield.



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